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D-component rotators as the classical limit of quantum SO(D) vector models

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Abstract. We consider a class of spin systems whose single-site configuration space is an orbit of a representation of a compact Lie group G. For these models we get upper and lower bounds to the quantum partition function in terms of two classical partition functions. If a certain group-theoretic condition is satisfied, then these bounds allow us to prove the convergence of a suitable sequence of quantum partition functions to the 'corresponding' classical one. This condition is shown to be satisfied, in particular, for the D-component rotators when D is odd. Our result could be useful for the extension of the Lee-Yang theorem to such models.

1. Introduction

In 1973 Lieb [1] obtained upper and lower bounds to the partition function of the quantum three-component rotator (quantum Heisenberg model), in terms of the partition function of the corresponding classical system. By means of these bounds it is possible to prove the convergence of the quantum partition function (or the free energy) to the classical one in the limit of *infinite angular momentum.*

D-component rotators are systems of interacting D-dimensional vectors of unit norm ($\|\varphi^\alpha\| = 1$) on a lattice Λ with N sites. Their statistical behaviour is determined by the following partition function:

$$Z(\rho, D) \equiv \int_{(S^{D-1})^N} \exp[-H(\rho\varphi)] \prod_{\alpha\in\Lambda} \mathrm{d}\nu(\varphi^\alpha) \qquad \varphi \in (S^{D-1})^N \qquad \rho \in \mathbb{R} \tag{1.1}$$

where S^{D-1} is the $(D-1)$-dimensional unit sphere considered as embedded in $\mathbb{R}^D$, and $\mathrm{d}\nu$ is the rotation-invariant measure on the sphere. The Hamiltonian H is assumed to be linear in the variables at each site. A common example is

$$-H(\varphi) \equiv \sum_{\substack{\alpha,\beta\in\Lambda \\ \alpha\neq\beta}} J^{\alpha\beta} \sum_{i=1}^{D} \varphi_i^\alpha \varphi_i^\beta + \sum_{\alpha\in\Lambda} \sum_{i=1}^{D} h_i^\alpha \varphi_i^\alpha. \tag{1.2}$$

The constant ρ which multiplies each spin has been introduced in (1.1) for further convenience. In the case $D = 3$ the quantum partition function is defined as follows. Let T be a representation of SO(3) on the vector space V (V is the single-spin quantum space) and let t be the corresponding 'infinitesimal' representation of the Lie algebra so(3). Replace now in the classical partition function each variable φ_i^α with the operator

$$t^\alpha(S_i) \equiv \mathbb{1}_1 \otimes \cdots \otimes \mathbb{1}_{\alpha-1} \otimes t(S_i) \otimes \mathbb{1}_{\alpha+1} \otimes \cdots \otimes \mathbb{1}_N$$

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where $\{S_i\}_{i=1}^3$ is a basis of so(3). $t^\alpha(S_i)$ acts on the tensor product of N copies of V. The integral over the classical configurations is also replaced by the trace of the resulting operator. So we define the quantum partition function as

$$Z^Q(\rho, \mathrm{SO}(3), k) \equiv (2k+1)^{-N} \operatorname{Tr} \exp[-H^Q(\rho t_k^\alpha(S_i))]. \tag{1.3}$$

k is an integer (or half integer if one considers SU(2)) which labels the irreducible representations of SO(3), and t_k is the $(2k+1)$-dimensional representation.

Lieb's result states, for each k, that

$$Z(k, 3) \leqslant Z^Q(1, \mathrm{SO}(3), k) \leqslant Z(k+1, 3) \tag{1.4}$$

so that, with a suitable rescaling of the parameters,

$$Z(1,3) \leqslant Z^Q(k^{-1}, \mathrm{SO}(3), k) \leqslant Z((k+1)/k, 3).$$

From this it follows that

$$Z(1,3) = \lim_{k\to\infty} Z^Q(k^{-1}, \mathrm{SO}(3), k).$$

This is the fundamental result which gives the classical partition function as the infinite-angular-momentum limit of the quantum partition function.

Besides the intrinsic interest, Lieb's limit theorem, together with the results of Asano [2, 3], Suzuki and Fisher [4], has allowed Dunlop and Newman [5] to prove the Lee–Yang theorem for those vector models which can be well approximated by classical ferromagnetic D-component rotators when $D = 2, 3$. In this important class of statistical systems there are, for instance, the vector models of φ^4 type, i.e. those with a single-site measure given by

$$\mathrm{d}\mu_0(\varphi^\alpha) = \exp(-\lambda \|\varphi^\alpha\|^4 + \gamma \|\varphi^\alpha\|^2) \qquad \varphi^\alpha \in \mathbb{R}^D.$$

A spin system is said to have the Lee–Yang property if its partition function, as a function of the complex magnetic field, is non-zero in a volume-independent region of the complex plane which contains the positive real axis. A consequence of the Lee–Yang property is the absence of phase transitions when the (real) magnetic field is non-zero.

Whether or not the Lee–Yang theorem holds for D-component rotators when $D > 3$ is still an open question. An important step in this direction could be an extension of Lieb's result to general D.

In this paper we show how such an extension can be carried out for an arbitrary odd $D = 2n+1$. It turns out that it is still possible to construct a quantum partition function which, in a certain limit, recovers the classical one. When $D = 3$ this limit reduces to the ordinary limit of infinite angular momentum. In the rest of the paper we will mostly deal with general spin systems in which the single-site configuration space is an orbit of a representation of a compact Lie group G. However in this introduction, for simplicity, we prefer to sketch the case $\mathrm{G} = \mathrm{SO}(D)$, which, having spherical orbits (in the standard representation), is connected with D-rotators.

We started from the reasonable expectation that, analogously to the $D = 3$ case, the quantum D-rotator could be defined by means of operators acting on some $\mathrm{SO}(D)$ representation space in such a way that (1.4) can be generalised to

$$Z(\beta_1(T), D) \leqslant Z^Q(1, \mathrm{SO}(D), T) \leqslant Z(\beta_2(T), D) \tag{1.5}$$

where T now denotes any representation of SO(D), which can no longer be (conveniently) specified with a single integer k, and β_1, β_2 are real numbers which depend on the representation T. If a sequence $\{T_k\}$ of representations of SO(D) exists such that

$$\lim_{k\to\infty} \frac{\beta_1(T_k)}{\beta_2(T_k)} = 1 \tag{1.6}$$

then we obtain the desired result

$$Z(1, D) = \lim_{k\to\infty} Z^Q(\beta_1(T_k)^{-1}, \mathrm{SO}(D), T_k). \tag{1.7}$$

The problem is how to define a SO(D) quantum model which can be considered as the quantum version of the D-component classical rotator and such that (1.5) and (1.6) hold. This is a non-trivial generalisation since, in the $D=3$ case, one exploits the correspondence

$$\{\varphi_i\}_{i=1}^3 \leftrightarrow \{S_i\}_{i=1}^3$$

between $\mathbb{R}^3$ and so(3). One cannot hope, of course, to extend this correspondence to the case $D>3$, since $\dim \mathrm{so}(D) = \frac{1}{2}D(D-1)$, which is, in general different from D. In other words, the operators $\{S_{ij}\}_{i<j}$, $i, j = 1, \ldots, D$, that form a basis for so(D), are rank-two antisymmetric tensors, while the elements of $\mathbb{R}^D$ are vectors. $D=3$ is a special situation just because, in that case, the two notions coincide.

In this paper we take a point of view which is in some sense complementary to that of Fuller and Lenard [6] and of Simon [7]. They *define* the quantum partition function of the quantum SO(D) model (Simon treats the more general case of any compact group) as

$$Z^Q(\rho, \mathrm{SO}(D), T) \equiv (\dim V)^{-N} \operatorname{Tr} \exp[-H^Q(\rho t(S_{ij}))] \tag{1.8}$$

and ask themselves what kind of models are recovered in the classical limit (1.7). Unfortunately in this way one never gets D-spheres as classical-limit manifolds (with $D>2$), so that (1.8) cannot be considered a good quantum version of the classical D-rotator. Actually Simon shows how classical $(D-1)$-spheres, for D odd, can be still obtained in a trickier indirect way. He associates the classical D-component rotator to the quantum SO($D+1$) model defined in (1.8), by means of the correspondence

$$\{\varphi_i\}_{i=1}^D \leftrightarrow \{S_{1i}\}_{i=2}^{D+1}. \tag{1.9}$$

This idea is successful because of the existence of a classical limit manifold which is a fibre bundle over S^{D-1}. However it has the drawback of requiring in (1.9) the choice of an arbitrary D-dimensional subspace of so($D+1$). Doing that, he explicitly breaks the rotational symmetry in the quantum model, and this can obscure some fundamental geometric properties of the system.

In the approach we present in this paper, on the contrary, we *impose* that the model we have to recover in the classical limit must be the D-rotator and we look for a suitable quantum SO(D) vector model that can accomplish this task. Of course our definition has to be as natural as possible and, when $D=3$, it must coincide with (1.3). The key idea is the following: since the classical variables of the D-rotator are vectors in $\mathbb{R}^D$, the variables of the corresponding quantum models must be linear operators which transform as a vector under SO(D). This means that, given a representation T

of SO(D) on a linear space V, we have to look for a set of linear operators $\{P_i\}_{i=1}^{D}$ on V, such that

$$T(R)^{-1}P_iT(R)=\sum_{j=1}^{D} R_{ij}P_j \qquad \forall R\in \mathrm{SO}(D) \tag{1.10}$$

where R_{ij} are the matrix elements of an element of SO(D). The quantum partition function is thus defined as

$$Z^Q(\rho, \mathrm{SO}(D), T)\equiv(\dim V)^{-N}\operatorname{Tr}\exp[-H^Q(\rho P_i)]$$

where the operators P_i depend, of course, on the representation T. By means of two basic inequalities of Berezin [8, 9] and Lieb [1] it is easy to see that condition (1.10) allows one to obtain bounds like (1.5). Unfortunately operators which satisfy (1.10) exist only when D is odd. Moreover, in that case they are not uniquely determined so that an ambiguity (besides the trivial multiplication of each P_i by a constant) seems to arise in the definition of the quantum partition function. This ambiguity can be conveniently resolved by demanding that the ratio $\beta_1(T)/\beta_2(T)$ is as suitable as possible (that is, near to 1) in view of (1.6). With this choice we will show how to find a sequence of representations T_k such that (1.6) is satisfied. In particular we get for SO($2n+1$)

$$\frac{\beta_1(T_k)}{\beta_2(T_k)}=\frac{k}{k+n}.$$

Our main result (corollary 3.6) is that, for each odd D, we have

$$\int_{(S^{D-1})^N}\exp[-H(\varphi)]\prod_{\alpha\in\Lambda}\mathrm{d}\nu(\varphi^\alpha)=\lim_{k\to\infty}(\dim V^{(k)})^{-N}\operatorname{Tr}\exp[-H^Q(P^{(k)})].$$

We also give an explicit expression for the representations T_k, as well as for the corresponding operators $P_i^{(k)}$.

2. Basic definitions and notation†

Throughout this paper $\Lambda=\{1,\ldots,N\}$ is a finite set which represents the lattice. We do not impose any geometric structure on Λ because it does not play any role. $N=|\Lambda|$ is the number of sites.

Given a finite-dimensional vector space Z, we denote by $L(Z)$ the set of linear operators on Z, and by GL(Z) the group of the invertible operators. $\mathbb{1}_Z$ is the identity operator. $\bar{x}$ denotes the complex conjugate of x, and A^* stands for the adjoint operator with respect to some scalar product.

G is a connected, semisimple, compact Lie group and $\mathrm{L_G}$ is its Lie algebra. In the following we will consider two representations $R:\mathrm{G}\mapsto \mathrm{GL}(W)$ and $T:\mathrm{G}\mapsto\mathrm{GL}(V)$ which act respectively on the vector spaces W and V. Both R and T are assumed to be continuous and irreducible. As a consequence they are also finite dimensional and unitary with a suitable choice of a scalar product in W and V. The corresponding 'infinitesimal' representations of $\mathrm{L_G}$ are denoted by r and t.

† Some notation used in the introduction can slightly differ from those of the rest of the paper, which are fixed in this section.

By the standard representation of a classical group (SU(n), SO(n), or Sp(n)) we mean the fundamental n-dimensional representation which simply operates on $\mathbb{C}^n$ by matrix multiplication.

We use either R_g or $R(g)$ to indicate the element $g \in \mathrm{G}$ represented in GL(W) and analogously for T.

2.1. *The classical partition function*

Even though we are principally interested in D-components rotators, and our sharpest results are limited to this case, our discussion can be carried out in a more general setup. This can be useful, in our opinion, for a deeper understanding of the problem. So we define a quite general family of classical spin systems which are 'indexed' by an orbit of an irreducible representation R of the Lie group G, acting on a vector space W of dimension D. The orbit, which is a subset of W, represents the single-spin classical configuration space. D-rotators are recovered, as we will see, by setting $\mathrm{G} = \mathrm{SO}(D)$ and taking R as the standard representation.

The Hamiltonian of the system is defined as a function $H: W^N \mapsto \mathbb{R}$, which can be expressed as a sum of monomials in the variables

$$x_i^\alpha, \bar{x}_i^\alpha \qquad \alpha \in \Lambda \qquad i = 1, \ldots, D$$

such that each monomial is of degree 1 in the variables at each site. A common example is given by (1.2).

The representation R naturally extends to a representation $\hat{R}$ of G^N on W^N setting, for each $x = (x^1, \ldots, x^N) \in W^N$,

$$\hat{R}_g x \equiv (R(g^1)x^1, \ldots, R(g^N)x^N)$$

with $g \equiv \{g^\alpha\}_{\alpha \in \Lambda}$. Now let x be an arbitrary element of W and let

$$\hat{x} \equiv (x, \ldots, x) \in W^N.$$

The classical partition function of the system is defined as

$$Z(R, x) \equiv \int_{\mathrm{G}^N} \exp[-H(\hat{R}_g \hat{x})] \prod_{\alpha \in \Lambda} \mathrm{d}\mu(g^\alpha) \qquad x \in W. \tag{2.1}$$

$\mathrm{d}\mu$ is the invariant (Haar) measure on G which exists because G is compact and thus unimodular [10]. Expression (2.1) can be given a more convenient form. Let, in fact,

$$K_x \equiv \{g \in \mathrm{G} | R_g x = x\}$$

be the isotropy group at x. The integration over the group G can be 'subdivided into' integrations over each left coset [11], i.e. unique left-invariant measure $\mathrm{d}\nu$ exists on the quotient space G/K_x such that for any continuous function f we have

$$\int_{\mathrm{G}} f(g)\, \mathrm{d}\mu(g) = \int_{\mathrm{G}/K_x} \left(\int_{K_x} f(gk)\, \mathrm{d}\mu_K(k) \right) \mathrm{d}\nu(gK_x)$$

where $\mathrm{d}\mu_K$ is the Haar measure on K_x. In the case of (2.1) the integrand is constant on each coset, and can be brought outside the inner integral. So if $p = gK_x$ is a generic left coset, we can symbolically set $p \cdot x \equiv R_g x$, and we set also

$$\hat{p} \cdot \hat{x} \equiv (p^1 \cdot x, \ldots, p^N \cdot x) \qquad \hat{p} \in (\mathrm{G}/K_x)^N.$$

With the normalisation $\int_{K_x} d\mu_K = 1$, we get

$$Z(R,x) = \int_{(G/K_x)^N} \exp[-H(\hat{p}\cdot\hat{x})] \prod_{\alpha\in\Lambda} d\nu(p^\alpha).$$

The coset space G/K_x is naturally homeomorphic to the orbit of x

$$\mathrm{Orb}(x) \equiv \{R_g x | g \in G\} \qquad x \in W$$

the homeomorphism being realised by the mapping $G/K_x \mapsto \mathrm{Orb}(x)$, $gK_x \mapsto R_g x$. This map induces a measure on $\mathrm{Orb}(x)$, so that (2.1) can be finally written as

$$Z(R,x) = \int_{\mathrm{Orb}(x)^N} \exp[-H(\varphi)] \prod_{\alpha\in\Lambda} d\nu(\varphi^\alpha). \tag{2.2}$$

We thus see that $Z(R,x)$ actually depends only on the orbit of R through x.

Example. Let $G = SO(D)$ with R the standard representation. In this case

$$W = \mathbb{R}^D \qquad K_x = SO(D-1) \qquad G/K_x = S^{D-1}$$

and

$$\mathrm{Orb}(x) = \|x\| S^{D-1} \equiv \{y \in \mathbb{R}^D | \, \|y\| = \|x\|\}.$$

Equation (2.2) becomes

$$Z(R,x) = \int_{(S^{D-1})^N} \exp[-H(\|x\|\varphi)] \prod_{\alpha\in\Lambda} d\nu(\varphi^\alpha) \tag{2.3}$$

where $d\nu$ is the rotation-invariant measure on the sphere. This is the usual partition function for the D-component rotator.

2.2. *The quantum partition function*

For the definition of the quantum partition function we need the following ingredients.

(i) A vector space V which plays the role of the single-spin quantum configuration space (the analogue of $\mathbb{C}^{2k+1}$ when $G = SO(3)$) and determines the quantum configuration space of the whole system as the tensor product

$$\hat{V} \equiv \bigotimes_{\alpha\in\Lambda} V^\alpha$$

where each V^α is an isomorphic copy of V.

(ii) A continuous, irreducible and unitary representation T of G acting on V.

(iii) A correspondence rule which associates a quantum Hamiltonian H^Q to a given classical Hamiltonian H. H^Q must be a self-adjoint linear operator on $\hat{V}$.

A natural correspondence rule can be constructed starting from the following consideration: since a *classical* spin is a vector in W, then a *quantum* spin must be defined as D-ple of operators $\{P_i\}_{i=1}^D$ acting on V, which transform under G as the components of a vector in the representation R. The following definition will sharpen this idea.

Definition 2.1. Let $\{w_i\}_{i=1}^D$ be a given basis of W and let $R_{ij}(g)$ be the matrix elements of R_g in this basis. A contravariant tensor operator (with respect to R, T and $\{w_i\}_{i=1}^D$) is a collection of operators $\{P_i\}_{i=1}^D$, where $P_i \in L(V)$, such that

$$T_g^{-1} P_i T_g = \sum_{j=1}^{D} R_{ij}(g) P_j \qquad \forall g \in G. \tag{2.4}$$

Condition (2.4) can be converted to infinitesimal form. Let r and t be the representations of L_G associated, respectively, with R and T. Every $g \in G$ contained in some neighbourhood of the identity can be expressed as $g = \exp A$ for some $A \in L_G$, where $\exp: L_G \to G$ is the standard exponential mapping. Moreover

$$R(\exp A) = \exp r(A) \qquad \forall A \in L_G$$

(the same applies to T, t). This equation, combined with (1.4), implies

$$[P_i, t(A)] = \sum_{j=1}^{D} r_{ij}(A) P_j \qquad \forall A \in L_G.$$

Example. Let $\{E_i\}$ be a basis of L_G. An important example of a tensor operator can be constructed setting $W \equiv L_G$ and

$$P_i \equiv t(E_i).$$

In fact, if $\mathrm{ad}: L_G \mapsto L(L_G)$ denotes the adjoint representation defined by

$$\mathrm{ad}_A(B) \equiv [A, B] \qquad A, B \in L_G$$

we get

$$[P_i, t(A)] = [t(E_i), t(A)] = -t(\mathrm{ad}_A E_i) = \sum_j \mathrm{ad}_{ji}(-A) t(E_j).$$

Thus the set $\{t(E_i)\}$ is a contravariant tensor operator with respect to the representation $r_{ij}(A) \equiv \mathrm{ad}_{ji}(-A)$, which in global form corresponds to

$$R_{ij}(g) \equiv \mathrm{Ad}_{ji}(g^{-1}). \tag{2.5}$$

The representation R is said to be contragredient with respect to the adjoint representation.

This representation seems to be in a certain sense privileged, because we can construct a 'standard' contravariant tensor operator starting from a basis of the Lie algebra of the group G. Unfortunately the choice (2.5) has a drawback: we know from (2.2) that the classical spin space turns out to be an orbit of the representation R. The trouble is that the adjoint representation (or its contragredient) never yields spherical orbits (with the exception of S^2) [7], ruling out the possibility of getting the classical rotators with more than three components in a *direct* way.

In the approach we present in this paper we choose R as the standard representation of SO(D). In this case the problem will be how to construct the appropriate contravariant tensor operator which gives the desired classical limit.

Once we have a contravariant tensor operator, we built up the quantum Hamiltonian as follows. Let $\{P_i\}_{i=1}^{D}$ be a contravariant tensor operator and let, for each $\alpha \in \Lambda$,

$$P_i^\alpha \equiv (\mathbb{1})_1 \otimes \cdots \otimes (\mathbb{1})_{\alpha-1} \otimes P_i \otimes (\mathbb{1})_{\alpha+1} \otimes \cdots \otimes (\mathbb{1})_N \in L(\hat{V})$$

where we have set $\mathbb{1}_\alpha \equiv \mathbb{1}_{V^\alpha}$. We define the quantum Hamiltonian H^Q to be that operator one obtains by replacing, in the classical Hamiltonian, the variables x_i^α with the operators P_i^α (and the variables $\bar{x}_i^\alpha$ with the operators $(P_i^\alpha)^*$) acting on $\hat{V}$. If, for instance, H is given by (1.2), then we set

$$-H^Q(P) \equiv \sum_{\substack{\alpha,\beta \in A \\ \alpha \neq \beta}} J^{\alpha\beta} \sum_{i=1} P_i^\alpha P_i^\beta + \sum_{\alpha \in \Lambda} \sum_{i=1} h_i^\alpha P_i^\alpha.$$

Note that, since the Hamiltonian is assumed to be of degree 1 in the variables at each site, and since P_i^α commutes with P_j^β whenever $\alpha \neq \beta$, no ordering ambiguity arises in our substitution rule. Moreover H^Q is self-adjoint, because H is real.

The quantum partition function is then defined as

$$Z^Q(R, T, P) \equiv (\dim V)^{-N} \operatorname{Tr} \exp[-H^Q(P)].$$

The rest of the paper is devoted to showing how, at least in the case of $\mathrm{G}=\mathrm{SO}(2n+1)$ and R the standard representation (corresponding to the $(2n+1)$-components rotator), a sequence of representations T_k exists such that

$$Z(R, x)=\lim_{k\to\infty} Z^Q(R, T_k, P^{(k)})$$

if we make the right choice for the operators $P^{(k)}=\{P_i^{(k)}\}_{i=1}^D$. We also give an explicit expression for both T_k and $P^{(k)}$.

3. Berezin–Lieb inequalities

The basic tools for proving upper and lower bounds (1.5) (in a generalised version) are two inequalities independently obtained by Berezin [8, 9], and Lieb [1]. Our exposition will follow [7].

Definition 3.1. Let V be a finite-dimensional vector space and (X, Σ, μ) a measure space such that $\int_X \mathrm{d}\mu=1$. A family of coherent projections is a measurable map $\pi: X\to L(V)$ such that

(i) For each $x\in X$, $\pi(x)$ is an orthogonal projection onto a one-dimensional subspace of V

(ii)

$$\int_X \pi(x)\,\mathrm{d}\mu(x)=c\mathbb{1}_V \text{ for some } c\in\mathbb{R}. \tag{3.1}$$

(Note that, since V is finite dimensional, this integral is well defined.) The constant c is actually determined. In fact we have the following.

Proposition 3.1. If $\pi: X\to L(V)$ is a family of coherent projections, then

$$\int_X \pi(x)\,\mathrm{d}\mu(x)=(\dim V)^{-1}\mathbb{1}_V.$$

Proof. Taking the traces of both sides of (3.1) we get

$$1=\int_X \mathrm{d}\mu(x)=c\operatorname{Tr} V=c\dim V. \qquad \square$$

Simple proofs of the following inequalities can be found in [7].

Theorem 3.2 (first Berezin–Lieb inequality). Let A be a self-adjoint operator on V. If we set $a_{\mathrm{L}}(x)\equiv \operatorname{Tr}(A\pi(x))$, then $a_{\mathrm{L}}\in L^\infty(X, \mathrm{d}\mu)$, and

$$\int_X \exp[a_{\mathrm{L}}(x)]\,\mathrm{d}\mu(x)\leqslant(\dim V)^{-1}\operatorname{Tr}\exp A.$$

Theorem 3.3 (second Berezin–Lieb inequality). Let $a_{\mathrm{U}}(x)\in L^\infty(X, \mathrm{d}\mu)$, be real valued, and define the operator

$$A\equiv\dim V\int_X a_{\mathrm{U}}(x)\pi(x)\,\mathrm{d}\mu(x).$$

Then we have

$$(\dim V)^{-1}\operatorname{Tr}\exp A\leqslant\int_X \exp[a_{\mathrm{U}}(x)]\,\mathrm{d}\mu(x).$$

The two functions a_L and a_U are, respectively, called by Simon the lower symbol and the upper symbol of A. In [7] many properties of lower and upper symbols are discussed.

4. Upper and lower bounds and the classical limit

The inequalities stated in the previous section allow us to obtain upper and lower bounds to the quantum partition function in terms of two classical partition functions. The group G^N (together with its Haar measure) will play the role of the measure space X. The first step is the construction of a family of coherent projections $\pi : \mathrm{G} \to L(V)$, where now V is the quantum spin space. We remember that G is assumed to act on V through the representation T. A scalar product $\langle \ , \ \rangle$ is introduced in V such that T is unitary.

Proposition 4.1. Let $u \in V$ be a fixed unit vector, and for each $g \in \mathrm{G}$ let $\pi_u(g)$ be the orthogonal projection onto the one-dimensional subspace generated by $T_g u$

$$\pi_u(g) v \equiv \langle T_g u, v \rangle T_g u.$$

Then π_u is a family of coherent projections on (G, μ), where μ is the invariant measure on G.

Proof. Since T is a continuous representation, π_u is continuous and, by consequence it is measurable. Moreover, we have

$$\pi_u(g) = T_g \pi_u T_g^{-1} \qquad \forall g \in \mathrm{G}$$

where $\pi_u \equiv \pi_u(e)$ and e is the unit element of G. If we define $M_u \in L(V)$ as

$$M_u \equiv \int_G \pi_u(g)\, \mathrm{d}\mu(g),$$

we obtain, $\forall h \in \mathrm{G}$,

$$T(h) M_u = \int_G T(hg) \pi_u T(g^{-1})\, \mathrm{d}\mu(g) = \int_G T(k) \pi_u T(k^{-1}h)\, \mathrm{d}\mu(k) = M_u T(h)$$

where we have set $k = hg$ and we have exploited the invariance of the measure. Thus M_u commutes with each operator T_h. Since T is irreducible, by Schur's lemma, $M_u = c\mathbb{1}_V$, and, by proposition 3.1

$$M_u = (\dim V)^{-1} \mathbb{1}_V. \qquad \square$$

We can now easily construct a family of coherent projections on G^N. In fact, given $u = \{u^\alpha\}_{\alpha \in \Lambda} \in V^N$, such that $\|u^\alpha\| = 1$, it is straightforward to verify that the map

$$\hat{\pi}_u : \mathrm{G}^N \mapsto L(\hat{V})$$

given by

$$\hat{\pi}_u(g) \equiv \bigotimes_{\alpha \in \Lambda} \pi_{u^\alpha}(g^\alpha) \qquad g = \{g^\alpha\}_{\alpha \in \Lambda} \tag{4.1}$$

is a family of coherent projections.

In order to apply the second Berezin-Lieb inequality to our quantum Hamiltonian we have to construct the upper symbol of a contravariant tensor operator. This is done by means of the following.

Theorem 4.2. Let $u \in V$ be the highest-weight vector of unit norm, relative to the representation T, and let $\{w_i\}_{i=1}^D$ a basis of W. A collection of operators $P = \{P_i\}_{i=1}^D$ is a contravariant tensor operator with respect to R, T and $\{w_i\}_{i=1}^D$ if and only if a vector $y \in W$ exists such that

$$P_i = \dim V \int_G (R_g y)_i \pi_u(g)\, d\mu(g) \qquad i = 1, \ldots, D \tag{4.2}$$

where $(R_g y)_i \equiv \langle w_i, R_g y \rangle$. Furthermore, if P is a contravariant tensor operator and

$$\langle u, P_i u \rangle = 0 \qquad i = 1, \ldots, D \tag{4.3}$$

then $P_i = 0$ for each i.

Proof. The *if* part of the first statement is straightforward. In fact, if (4.2) holds, then

$$\begin{aligned} T_h^{-1} P_i T_h &= \dim V \int_G (R_g y)_i \pi_u(h^{-1}g)\, d\mu(g) \\ &= \dim V \int_G (R_{hg} y)_i \pi_u(g)\, d\mu(g) \\ &= \sum_{j=1}^D R_{ij}(h) P_j. \end{aligned}$$

The *only if* part, which is much less trivial, follows from theorem A.2.3 of [7] (we adopt a different convention for tensor operators so that our matrix R is the complex conjugate of his matrix V). There it is proved that each set of operators $\{P_i\}_{i=1}^D$ satisfying (2.4) has the form

$$P_i = \int_G f_i(g) \pi_u(g)\, d\mu(g)$$

where the functions f_i can be chosen in such a way that

$$f_i(gh) = \sum_{j=1}^D R_{ij}(g) f_j(h) \qquad \forall g, h \in \mathrm{G}.$$

Setting $y_i \equiv (\dim V)^{-1} f_i(e)$ we get

$$f_i(g) = \dim V \sum_{j=1}^D R_{ij}(g) y_j = \dim V (R_g y)_i$$

from which (4.3) follows.

The second statement follows from lemma A.2.1 of [7]. We have only to note that (2.4) and (4.3) imply

$$\mathrm{Tr}(P_i \pi_u(g)) = \langle T_g u, P_i T_g u \rangle = 0 \qquad \forall g \in \mathrm{G}, \qquad i = 1, \ldots, D. \qquad \square$$

Since each contravariant tensor operator has the form (4.2), from now on we denote by $P(y)$ the operator associated with y, i.e.

$$P(y)_i \equiv \dim V \int_G (R_g y)_i \pi_u(g)\, d\mu(g) \qquad i = 1, \ldots, D. \tag{4.4}$$

The vector u which appears on the RHS of (4.4) is understood to be the unit-norm highest-weight vector for the representation T. Our first general result is the following.

Theorem 4.3. Let R and T be two representations of G satisfying the hypotheses of section 2, and let $u \in V$, $\|u\| = 1$, be the highest-weight vector for T. For each $y \in W$ we have

$$Z(R, x) \leq Z^Q(R, T, P(y)) \leq Z(R, y)$$

where the operators $P(y)_i$ are given by (4.4) and $x \in W$ is defined as

$$x_i \equiv \langle u, P(y)_i u\rangle \qquad i = 1, \ldots, D.$$

Proof. Let $\hat{x}, \hat{y} \in W^N$ be given by

$$\hat{x} \equiv (x, \ldots, x) \qquad \hat{y} \equiv (y, \ldots, y).$$

Looking at (2.1) and remembering that H is real and H^Q is self-adjoint, it is clear that if we prove that $H(\hat{R}_g\hat{x})$ and $H(\hat{R}_g\hat{y})$ are respectively the lower and the upper symbol of $H^Q(P(y))$ with respect to the family of coherent projections (4.1), then this theorem is a consequence of the two Berezin–Lieb inequalities. So we must show that

$$\mathrm{Tr}[H^Q(P(y))\hat{\pi}_u(g)] = H(\hat{R}_g\hat{x}) \qquad \forall g \in \mathrm{G}^N \tag{4.5}$$

and that

$$H^Q(P(y)) = \dim \hat{V} \int_{\mathrm{G}^N} H(\hat{R}_g\hat{y})\hat{\pi}_u(g) \prod_{\alpha \in \Lambda} \mathrm{d}\mu(g^\alpha). \tag{4.6}$$

The second equation follows easily from (4.4). Furthermore, for each $g \in \mathrm{G}$ we have

$$\begin{aligned}
\mathrm{Tr}[P(y)_i\pi_u(g)] &= \langle T_g u, P(y)_i T_g u\rangle \\
&= \langle u, T_g^{-1} P(y)_i T_g u\rangle \\
&= \sum_{j=1}^{D} R_{ij}(g)\langle u, P(y)_j u\rangle \\
&= \sum_{j=1}^{D} R_{ij}(g) x_j \\
&= (R_g x)_i
\end{aligned}$$

and, analogously,

$$\mathrm{Tr}[(P(y)_i)^* \pi_u(g)] = (\bar{R}_g \bar{x})_i.$$

The quantum Hamiltonian is a sum of terms which have the form

$$\bigotimes_{\alpha \in \Delta} P(y)^\alpha_{i_\alpha} \bigotimes_{\beta \in \bar{\Delta}} (P(y)^\beta_{i_\beta})^* \bigotimes_{\gamma \in \Lambda \setminus (\Delta \cup \bar{\Delta})} (\mathbb{1})_\gamma$$

where Δ and $\bar{\Delta}$ are two disjoint subsets of Λ. Since $\mathrm{Tr}(A \otimes B) = \mathrm{Tr}\, A\, \mathrm{Tr}\, B$, taking the trace of $H^Q(P(y))\hat{\pi}_u(g)$ is equivalent to replacing in H^Q each $P(y)^\alpha_i$ with $(R(g^\alpha)x^\alpha)_i$, and each $(P(y)^\alpha_i)^*$ with $(\bar{R}(g^\alpha)\bar{x}^\alpha)_i$. Thus (4.5) follows. □

4.1. The classical limit

A straightforward consequence of theorem 4.3 is the following classical-limit result.

Theorem 4.4. Assume that you can find a sequence of representations $T_k: \mathrm{G} \mapsto \mathrm{GL}(V_k)$, and a vector $y \in W$ such that,

$$x^{(k)} \equiv \langle u_k, P^{(k)}(y) u_k \rangle \overset{k\to\infty}{\longrightarrow} y \tag{4.7}$$

where

$$P^{(k)}(y)_i \equiv \dim V_k \int_G (R_g y)_i \pi_{u_k}(g)\, \mathrm{d}\mu(g) \tag{4.8}$$

and u_k is a unit highest-weight vector in V_k. Then

$$Z(R, y) = \lim_{k\to\infty} Z^Q(R, T_k, P^{(k)}(y)) \tag{4.9}$$

follows.

Of course one could hope that a result like (4.9) would hold for any 'reasonable' choice of G and R. This is unfortunately false. In the next section, in fact, we prove that a necessary condition for getting

$$\langle u, P(y) u \rangle \neq 0$$

(which is in turn necessary for having $P(y) \neq 0$) is that y must belong to the zero-weight subspace of W. This excludes all the representations R which do not have the zero weight. Among these there is also the standard representation of SO($2n$) which would yield the ($2n$)-components rotator. In other words, if G = SO($2n$) there are no non-trivial contravariant tensor operators with respect to R and T, when R is the standard representation.

Our results are for this reason limited to rotators with an odd number of components. In this case the standard representation has a one-dimensional zero-weight subspace, and we will prove that condition (4.7) can be satisfied with an appropriate choice of T_k. We recall here for clarity (see the next section for more details) that the finite-dimensional irreducible representations of SO($2n+1$) are identified by a non-increasing n-ple of non-negative integers

$$(k_1 \geqslant k_2 \geqslant \ldots \geqslant k_n)$$

which are the components of highest weight in a suitable basis. For those who are familiar with Young-tableau methods, we say that k_i represents the length of the ith row of the Young tableau associated with the representation.

Theorem 4.5. Let G = SO(D), $D = 2n+1$, let R be the standard representation of SO($2n+1$) acting on $W = \mathbb{R}^D$ and let T_k be the representation of SO($2n+1$) with highest weight $(k, \ldots, k)$. If $P^{(k)}(y)$ is the associated contravariant tensor operator given by (4.8), where y is an element of the (one-dimensional) zero-weight subspace of W, then

$$x^{(k)} = \frac{k}{k+n} y$$

so that (4.7) holds.

We give the proof in the next section. As we have already remarked, this result implies that the classical partition function of D-rotators can be obtained as a kind of 'infinite angular momentum limit' of the quantum one.

Corollary 4.6. With the hypotheses of theorem 4.5 we have, for D odd,

$$\int_{(S^{D-1})^N} \exp[-H(\|y\|\phi)] \prod_{\alpha\in\Lambda} \mathrm{d}\nu(\varphi^\alpha) = \lim_{k\to\infty} (\dim V_k)^{-N} \operatorname{Tr}\exp[-H^Q(P^{(k)}(y))].$$

Proof. It follows from theorems 4.4, 4.5 and from (2.3). □

5. Proof of theorem 4.5

In this section we will show how to compute, in the case of D-component rotators the quantities

$$x_i \equiv \langle u, P(y)_i u\rangle \qquad i = 1, \ldots, D$$

where u is the highest-weight vector for a certain representation T of $\mathrm{SO}(D)$, and $P(y)_i$ are given by (4.4).

We briefly recall some known facts about Lie algebras and their representations in order to fix our notation. See, for instance, [12] for an extensive discussion.

5.1. The structure of a compact semisimple Lie algebra L_G

First we introduce the complex extension $(\mathrm{L_G})_c$ of $\mathrm{L_G}$. The motivation for dealing with $(\mathrm{L_G})_c$ is that in general it is easier to study complex rather than real Lie algebras. On the other hand the irreducible representations of $(\mathrm{L_G})_c$ are just the complexified irreducible representations of $\mathrm{L_G}$ [12, chap X.14], thus it is possible to get information about the irreducible representations of the real Lie algebra $\mathrm{L_G}$, exploiting the properties of $(\mathrm{L_G})_c$.

Fix in $(\mathrm{L_G})_c$ a Cartan subalgebra H (unfortunately H is the traditional symbol for both the Hamiltonian and the Cartan subalgebra; we hope that no confusion will arise) and denote its dual by H^*. If $\Delta \subset \mathrm{H}^*$ is the set of non-zero roots relative to H, we have

$$(\mathrm{L_G})_c = \mathrm{H} + \sum_{\alpha\in\Delta} L_\alpha$$

where L_α is the one-dimensional root space. Thus we have

$$[X, Y_\alpha] = \alpha(X) Y_\alpha \qquad \forall X \in \mathrm{H} \qquad \forall Y_\alpha \in L_\alpha a.$$

For each $\alpha \in \Delta$ an unique element $X_\alpha \in \mathrm{H}$ exists such that

$$\alpha(X) = (X_\alpha, X) \qquad \forall X \in \mathrm{H}$$

where (,) is the Killing form on $(\mathrm{L_G})_c$. We also choose an element E_α in each L_α normalised in such a way that

$$[E_\alpha, E_{-\alpha}] = X_\alpha. \tag{5.1}$$

Now let $s : \mathrm{L_G} \mapsto \mathrm{L}(Z)$ be any irreducible representation of $\mathrm{L_G}$ on a finite-dimensional vector space Z, and let $\tilde{s} : (\mathrm{L_G})_c \mapsto \mathrm{L}(Z)$ be the complex extension of s, defined by

$$\tilde{s}(X + \mathrm{i}Y) \equiv s(X) + \mathrm{i}s(Y) \qquad X, Y \in \mathrm{L_G}.$$

The representation space Z can be decomposed as a sum of orthogonal weight subspaces

$$Z = \bigoplus_\lambda Z^{(\lambda)}$$

where

$$Z^{(\lambda)} \equiv \{z \in Z | \tilde{s}(X)z = \lambda(X)z, \forall X \in \mathrm{H}\}.$$

This decomposition remains the same if we consider the real weights, so we will not make any distinction between the two cases. An important property of the weight spaces is that if $\rho + \alpha$ is a weight, then

$$\tilde{s}(E_\alpha)Z^{(\rho)} \subset Z^{(\rho+\alpha)}. \tag{5.2}$$

Since $\mathrm{L_G}$ is compact, a complex scalar product $\langle\ ,\ \rangle$ can be chosen in Z in such a way that $s(A)$ is anti-Hermitean for each $A \in \mathrm{L_G}$. This is equivalent to requiring that the global representation S of G is unitary. With respect to such scalar product and with the normalisation (5.1), one gets

$$s(E_\alpha)^* = s(E_{-\alpha}) \qquad \forall \alpha \in \Delta \tag{5.3}$$

where A^* denotes the adjoint of A.

We anticipate here a result which will be useful in the following.

Proposition 5.1. If $z \in Z^{(\rho)}$ is a unit vector such that $s(E_\alpha)z = 0$, then

$$\|s(E_{-\alpha})z\|^2 = (\rho, \alpha) = \rho(X_\alpha).$$

Proof. The computation is straightforward. We just have to remember (5.1), (5.3) and the morphism property $s[A, B] = [s(A), s(B)]$. In fact

$$\begin{aligned}
\|s(E_{-\alpha})z\|^2 &= \langle s(E_{-\alpha})z, s(E_{-\alpha})z\rangle \\
&= \langle z, s(E_{-\alpha})^* s(E_{-\alpha})z\rangle \\
&= \langle z, s(E_\alpha)s(E_{-\alpha})z\rangle \\
&= \langle z, [s(E_\alpha), s(E_{-\alpha})]z\rangle + \langle z, s(E_{-\alpha})s(E_\alpha)z\rangle \\
&= \langle z, s(X_\alpha)z\rangle \\
&= \langle z, \rho(X_\alpha)z\rangle \\
&= \rho(X_\alpha).
\end{aligned}$$

□

5.2. *Proof of theorem 4.5*

We now return to our original problem of computing x_i. Setting

$$D_{ij} \equiv \dim V \int_G R_{ij}(g)|\langle u, T_g u\rangle|^2 \, \mathrm{d}\mu(g) \tag{5.4}$$

we have

$$x_i = \sum_{j=1}^{D} D_{ij} y_j. \tag{5.5}$$

Integrals of the type (5.4) can be expressed in terms of the so-called Clebsch-Gordan coefficients. The strategy is the following. Suppose, to be general, that Q is a third irreducible representation of G on the vector space Y and choose orthonormal bases $\{w_i\}$, $\{v_a\}$ and $\{y_p\}$ respectively in W, V and Y. We want to compute the quantities

$$\dim Y \int_G R_{ij}(g) T_{ab}(g) \bar{Q}_{pq}(g)\, \mathrm{d}\mu(g).$$

Consider the tensor product representation between R and T, defined by $R \otimes T: G \mapsto \mathrm{GL}(W \otimes V)$, $g \mapsto R_g \otimes T_g$, or, in infinitesimal form, $r \otimes t: \mathrm{L}_G \mapsto \mathrm{L}(W \otimes V)$, $X \mapsto r_X \otimes \mathbb{1}_V + \mathbb{1}_W \otimes t_X$. $R \otimes T$ can be decomposed as a sum of irreducible representations

$$R \otimes T = \bigoplus_\tau S_\tau. \tag{5.6}$$

This means that it is possible to write

$$W \otimes V = \bigoplus_\tau Z_\tau \tag{5.7}$$

where each Z_τ is an invariant subspace for $R \otimes T$ and the restriction of $R \otimes T$ to each subspace

$$S_\tau(g) \equiv (R \otimes T)_g]_{Z_\tau}$$

is an irreducible representation of G. For those S_τ which are equivalent to Q (we write $S_\tau \cong Q$) let U_τ be the unitary operators that realises the equivalence, i.e. $U_\tau: Y \mapsto Z_\tau$ are such that $S_\tau = U_\tau Q U_\tau^{-1}$. Then, by means of the orthogonality relations [10, chap. 7], it can be easily proved that

$$\dim Y \int_G R_{ij}(g) T_{ab}(g) \bar{Q}_{pq}(g)\, \mathrm{d}\mu(g) = \sum_{\tau: S_\tau \cong Q} \langle w_i \otimes v_a, U_\tau y_p \rangle \langle U_\tau y_q, w_j \otimes v_b \rangle.$$

In our case $Q = T$, hence (5.4) becomes

$$D_{ij} = \sum_{\tau: S_\tau \cong T} \langle w_i \otimes u, u_\tau \rangle \langle u_\tau, w_j \otimes u \rangle \tag{5.8}$$

where now $u_\tau \equiv U_\tau u \in Z_\tau$ is the highest-weight vector for the representation S_τ. So a first necessary condition in order to get $x \neq 0$ (and so $P(y) \neq 0$) is:

(c_1) the decomposition of the tensor prdouct $R \otimes T$ must contain a factor equivalent to T.

A second condition is easily found if we choose the basis $\{w_i\}_{i=1}^D$ in such a way that each element w_j has a definite weight. In fact, with this convention, we have the following result.

Proposition 5.2. The matrix element D_{ij} is non-zero only if w_i and w_j are both zero-weight vectors.

Proof. Clearly u_τ (for all τ such that $S_\tau \cong T$) has the same weight of u. Since

$$\mathrm{weight}(w_i \otimes u) = \mathrm{weight}(u) + \mathrm{weight}(w_i)$$

if w_i has a non-zero weight, then the $w_i \otimes u$ and u_τ have different weights and so they are orthogonal. From (5.8), $D_{ij} = 0$ follows. The same applies to w_j. □

This proposition suggests to us that the only good candidates are those representations R such that:

(c_2) the zero-weight space of $R(W^{(0)})$ is non-trivial.

Since we are mainly interested in the classical D-component rotators which are recovered by setting $G = SO(D)$ with R the standard representation, let us check what happens in this case. We discuss separately the cases D even and D odd, which have different characteristics.

5.2.1. The Lie algebra so(2n+1). The Lie algebra $so(2n+1, \mathbb{C})$ can be shown [12] to be isomorphic to the algebra of the matrices $\{A_{ij}\}$, $i, j = 0, \pm 1, \dots, \pm n$ which satisfy

$$A_{i,j} = -A_{-j,-i} \qquad i, j = 0, \pm 1, \dots, \pm n. \tag{5.9}$$

This representation is, in some respect, more convenient than the usual representation in term of anti-symmetric matrices, because, as we will see, it is possible to choose a Cartan subalgebra made up of diagonal matrices.

If we let e_{ij} be the matrix whose elements are given by $(e_{ij})_{kl} \equiv \delta_{ik}\delta_{jl}$, then the set

$$E_{ij} \equiv e_{ij} - e_{-j,-i} \qquad i, j = 0, \pm 1, \dots, \pm n \qquad i + j > 0 \tag{5.10}$$

is a basis of $so(2n+1, \mathbb{C})$. Of course, $\dim so(2n+1) = (2n+1)n$. The vector subspace of the diagonal matrices which satisfy (5.9) is a Cartan subalgebra H, and the elements $\{h_j\}_{j=1}^n$, $h_i \equiv E_{ii}$ form a basis of H. So $\dim H = n$. Let $\{\lambda_i\}_{i=1}^n$ be the usual basis in H* defined by

$$\lambda_i(h_j) = \delta_{ij}.$$

The Killing form can be normalised in such a way that

$$(h_i, h_j) = (\lambda_i, \lambda_j) = \delta_{ij}.$$

It is also convenient to set $h_{-i} \equiv -h_i$, $\lambda_{-i} \equiv -\lambda_i$ and $h_0 = \lambda_0 \equiv 0$. With this convention we have

$$[h_i, E_{jk}] = (\delta_{ij} + \delta_{i,-k} - \delta_{ik} - \delta_{i,-j}) E_{jk}$$

which is equivalent to

$$[h, E_{jk}] = (\lambda_j - \lambda_k)(h) E_{jk} \qquad \forall h \in H.$$

From these relations we can read directly the set of the roots of $so(2n+1, \mathbb{C})$:

$$\Delta = \{\alpha_{ij} = \lambda_i - \lambda_j | -n \leq i, j \leq n, \ i + j > 0, \ i \neq j\}.$$

The root spaces $L_{\alpha_{ij}}$ are spanned by the elements E_{ij}. Any irreducible representation of $so(2n+1)$ is determined by its highest weight. The set of the possible highest weights which correspond to a single-valued global representation of $SO(2n+1)$ is

$$\{k_1\lambda_1 + \dots + k_n\lambda_n | k_1 \geq k_2 \geq \dots \geq k_n \geq 0, \ k_i \text{ integers}\}.$$

Consider now the standard representation R which acts on $W = \mathbb{C}^{2n+1}$. Taking the standard basis $\{w_i\}_{i=-n}^n$ in W, given by

$$(w_i)_j \equiv \delta_{ij} \tag{5.11}$$

it can be checked that

$$Xw_i = \lambda_i(X) w_i \qquad \forall X \in H \qquad i = 0, \pm 1, \dots, \pm n. \tag{5.12}$$

In particular

$$Xw_0 = 0 \qquad \forall X \in \mathrm{H}.$$

Hence W has a zero-weight space of dimension 1 generated by w_0. Unfortunately the situation is different when D is even.

5.2.2. The Lie algebra so(2n). The Lie algebra so$(2n, \mathbb{C})$ is isomorphic to the algebra of the matrices $\{A_{ij}\}$, $i, j = \pm 1, \ldots, \pm n$ which satisfy

$$A_{i,j} = -A_{-j,-i} \qquad i, j = \pm 1, \ldots, \pm n.$$

So we can carry out the discussion in a similar way to the so$(2n+1)$ case, with the difference that the indices do not take the value zero. As it is easy to guess, (5.12) becomes

$$Xw_i = \lambda_i(X) w_i \qquad \forall X \in \mathrm{H} \qquad i = \pm 1, \ldots, \pm n.$$

This means that the set of the weights for the standard representation is

$$\{\pm\lambda_1, \ldots, \pm\lambda_n\}.$$

Hence this representation has no zero weight. This implies that

$$x_i \equiv \langle u, P(y)_i u \rangle = 0$$

and, by consequence, the possibility of getting a non-trivial quantum partition function is ruled out. On the contrary for odd D, we have the following lemma.

Lemma 4.3. Let $R : \mathrm{G} \mapsto \mathrm{GL}(W)$ be the standard representation of SO$(2n+1)$, with highest weight λ_1 and let $T : \mathrm{G} \mapsto \mathrm{GL}(V)$ be the representation whose highest weight is $k(\lambda_1 + \ldots + \lambda_n)$, where k is a positive integer. Then

(i) the tensor product $R \otimes T$ contains exactly one factor equivalent to T; more precisely (5.6) and (5.7) take the form

$$R \otimes T = S_1 \oplus S_2 \oplus S_3 \tag{5.13}$$

$$W \otimes V = Z_1 \oplus Z_2 \oplus Z_3 \tag{5.14}$$

where, denoting by ρ_i the highest weight of S_i, we have

$$\rho_1 = (k+1)\lambda_1 + k(\lambda_2 + \ldots + \lambda_n) = (k+1, k, \ldots, k)$$

$$\rho_2 = k(\lambda_1 + \ldots + \lambda_n) = (k, \ldots, k)$$

$$\rho_3 = k(\lambda_1 + \ldots + \lambda_{n-1}) + (k-1)\lambda_n = (k, \ldots, k, k-1)$$

(ii) let u be the highest-weight vector in V, $\hat{u}$ be the highest-weight vector in Z_2 and w_0 be the zero-weight vector in W. All these vectors are assumed of unit norm. Then

$$|\langle w_0 \otimes u, \hat{u} \rangle|^2 = \frac{k}{k+n}. \tag{5.15}$$

Proof. The rules for the decomposition of the tensor product of two representations of SO$(2n+1)$, and for the computation of the multiplicities of the weights, although straightforward, are not simple enough to be stated here. A systematic exposition can be found in [13, 14]. In particular, statement (i) follows from formula (1) which appears at the end of p 509 of [13], and from the rules explained in section 2.4 in the same paper.

To prove (ii) we use the following method. The problem is that of identifying $\hat{u}$ as an element in $W \otimes V$.

Equation (5.14), restricted to the weight-ρ subspace takes the form

$$(W \otimes V)^{(\rho)} = Z_1^{(\rho)} \oplus Z_2^{(\rho)} \oplus Z_3^{(\rho)}.$$

Since $\rho_1 > \rho_2 > \rho_3$, in particular we obtain

$$(W \otimes V)^{(\rho_2)} = Z_1^{(\rho_2)} \oplus Z_2^{(\rho_2)}.$$

But ρ_2 is the highest weight in Z_2, thus $Z_2^{(\rho_2)}$ is one dimensional. This means that we can identify $\hat{u}$ as any unit vector in the orthogonal complement of $Z_1^{(\rho_2)}$ considered as a subspace of $(W \otimes V)^{(\rho_2)}$. So the problem is solved if we find a basis for $Z_1^{(\rho_2)}$.

Let $\{w_i\}_{i=-n}^{n}$ be the basis of W defined in (5.11). The vector $w_1 \otimes u$, having weight ρ_1, necessarily belongs Z_1. Since Z_1 in an invariant subspace of $R \otimes T$, we know that all the vectors of the form

$$(r \otimes t)(A_1) \ldots (r \otimes t)(A_m) w_1 \otimes u \qquad A_i \in \mathrm{L_G}$$

still belong to Z_1. So we can find a basis of $Z_1^{(\rho_2)}$ acting on $w_1 \otimes u$ with appropriate strings of lowering operators $(r \otimes t)(E_\alpha)$.

It is convenient to define the following particular weights:

$$\sigma_j \equiv \rho_2 - \lambda_j = \sum_{\substack{i=1 \\ i \neq j}}^{n} k\lambda_i + (k-1)\lambda_j \qquad j = 1, \ldots, n.$$

Equation (5.15) is a consequence of the following three statements.

(a) The subspaces $V^{(\sigma_j)}, j = 1, \ldots, n$, are one dimensional.

(b) Choose a unit vector v_j in each $V^{(\sigma_j)}$. Then $\dim(W \otimes V)^{(\rho_2)} = n+1$, and the $n+1$ vectors

$$w_0 \otimes u \qquad \{w_j \otimes v_j\}_{j=1}^{n}$$

form an orthonormal basis of $(W \otimes V)^{(\rho_2)}$.

(c) Real numbers $\beta_1, \ldots, \beta_n$ exist such that the n vectors

$$w_0 \otimes u + \mathrm{e}^{\mathrm{i}\beta_j} \sqrt{k}\, w_j \otimes v_j \qquad j = 1, \ldots, n$$

form a basis of $Z_1^{(\rho_2)}$.

Proof of (ii), given (a), (b) and (c). From (b) and (c) we know that a unit vector orthogonal to $Z_1^{(\rho_2)}$ is given by

$$\hat{u} = \frac{1}{\sqrt{k+n}} \left(\sqrt{k}\, w_0 \otimes u - \sum_{j=1}^{n} \mathrm{e}^{-\mathrm{i}\beta_j} w_j \otimes v_j \right).$$

Hence we obtain

$$\langle w_0 \otimes u, \hat{u} \rangle = \left(\frac{k}{k+n} \right)^{1/2}.$$

Proof of (a). The statement follows from corollary 2 of [14].

Proof of (b). Since the weight of the tensor product is given by the sum of weights, we have, in general

$$(W \otimes V)^{(\rho)} = \bigoplus_{\substack{\mu, \nu \\ \mu + \nu = \rho}} W^{(\mu)} \otimes V^{(\nu)}.$$

Equation (5.12) says that the weight of w_i is λ_i, thus

$$W = \bigoplus_{j=-n}^{n} W^{(\lambda_j)}$$

and as a consequence,

$$(W \otimes V)^{(\rho_2)} = (W^{(0)} \otimes V^{(\rho_2)}) \oplus \left(\bigoplus_{i=1}^{n} W^{(\lambda_j)} \otimes V^{(\sigma_j)} \right) \oplus \left(\bigoplus_{j=1}^{n} W^{(-\lambda_j)} \otimes V^{(\rho_2+\lambda_j)} \right).$$

But $V^{(\rho_2+\lambda_j)} = \{0\}$ because $\rho_2 + \lambda_j$ is higher than the highest weight, ρ_2, in V. Thus

$$(W \otimes V)^{(\rho_2)} = (W^{(0)} \otimes V^{(\rho_2)}) \oplus \left(\bigoplus_{j=1}^{n} W^{(\lambda_j)} \otimes V^{(\sigma_j)} \right)$$

and since $V^{(\sigma_j)}$ are one dimensional, (b) follows.

Proof of (c). Let E_{ij} be the root vector of so$(2n+1)$ given by (5.10), corresponding to the root $\alpha_{ij} \equiv \lambda_i - \lambda_j$. Let $\Gamma_{ij} \equiv (r \otimes t)(E_{ij})$. Z_1 is an invariant subspace of $r \otimes t$, thus $\Gamma_{ij} z \in Z_1$ if $z \in Z_1$. In particular, if we set $z \equiv w_1 \otimes u \in Z_1$, the n vectors

$$u_1 \equiv \Gamma_{01} z \qquad u_j \equiv \Gamma_{0j} \Gamma_{j1} z \qquad j = 2, \ldots, n$$

are all contained in Z_1. Furthermore (5.2) implies that

$$\text{weight}(\Gamma_{ij} v) = \text{weight}(v) + \lambda_i - \lambda_j$$

thus we have weight$(u_j) = \rho_2$, so that $u_j \in Z_1^{(\rho_2)}$, $j = 1, \ldots, n$. We now want to show that

$$u_j = w_0 \otimes u + e^{i\beta_j} \sqrt{k}\, w_j \otimes v_j.$$

We need some preliminary calculations.

(I) For each $j = 2, \ldots, n$, weight$(t(E_{j1})u) = \rho_2 - \lambda_1 + \lambda_j$. From corollary 2 of [14], it follows that $V^{(\rho_2 - \lambda_1 + \lambda_j)} = \{0\}$. Hence

$$t(E_{j1}) u = 0.$$

(II) By proposition (5.1) we have

$$\| t(E_{0j}) u \|^2 = (\lambda_j, \rho_2) = k.$$

Moreover, weight$(t(E_{0j})u) = \sigma_j$. As a consequence

$$t(E_{0j}) u = e^{i\beta_j} \sqrt{k}\, v_j.$$

(III) Using the explicit expressions (5.10) and (5.11), we get

$$r(E_{0j}) w_j \equiv E_{0j} w_j = w_0 \qquad r(E_{j1}) w_1 \equiv E_{j1} w_1 = w_j.$$

We can now collect all pieces, and we get

$$\begin{aligned} u_j &= \Gamma_{0j} \Gamma_{j1} z = \Gamma_{0j} [r(E_{j1}) w_1 \otimes u + w_1 \otimes t(E_{j1}) u] \\ &= \Gamma_{0j} w_j \otimes u \\ &= w_0 \otimes u + e^{i\beta_j} \sqrt{k}\, w_j \otimes v_j. \end{aligned}$$

Thus we have obtained that the n vectors u_j are all contained in $Z_1^{(\rho_2)}$. Since they are also linearly independent this completes the proof. □

Lemma 5.3 and proposition 5.2 tell us that, in the standard basis $\{w_j\}_{j=-n}^{n}$ of W, defined in (5.11), the matrix element D_{ij} are given by

$$D_{00}=\frac{k}{k+n}$$

$$D_{ij}=0 \qquad \text{if } i\neq 0 \text{ or } j\neq 0$$

so, letting $y=w_0$, by (5.5), we obtain

$$x=\frac{k}{k+n}y$$

which proves theorem 4.5.

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